Real space renormalisation group method for the dilute Gaussian model and its relation to the random walk problems near the percolation threshold

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Abstract. An isomorphism between the Gaussian model in the magnetic field and the time-dependent conduction–diffusion problem permits us to develop a real space renormalisation group method for the diluted version of this model by which all single cluster results of the phenomenological scaling theory of Gefen, Aharony and Alexander are correctly reproduced. An extension of this model to the case of strong bias then provides some restrictions on the validity of their results. The result for the density of states, which we reproduce in the form predicted by Alexander and Orbach, is subject to the same kind of restrictions.

1. Introduction

A great deal of work recently has been devoted to the study of random walks on percolation and fractal structures. An excellent review of these efforts may be found in the September 1984 issue of *Journal of Statistical Mechanics* which is entirely devoted to the above problematics. It is believed (Coniglio 1983 and references therein) that cluster structure near the percolation threshold can be considered as a random fractal. We consider here an extension of the isomorphism between the Gaussian model in the constant magnetic field and the random walk problem to the case of diluted Gaussian model which corresponds to the symmetric random hopping on the above fractal percolative structure. Very simple real space renormalisation group (RSRG) analysis has enabled us to rigorously reproduce all the single-cluster scaling results of Gefen *et al* (1983) as well as the phenomenological results for the density of states proposed by Alexander and Orbach (1982). Our investigation demonstrates the rather non-trivial connection between the static and dynamic random walk exponents. Phenomenological scaling theories alone are unable to satisfactorily explain the source of anomalous behaviour of the random walk in the vicinity of percolation threshold. In spite of the apparent success of the above theories in terms of rather good numerical agreement with Monte Carlo data (Pandey *et al* 1984), their extensions to more complicated cases such as the inclusion of electron–electron interactions, strong external fields, etc, is rather formidable if not impossible. The formalism developed at the same time permits us to consider more complicated situations and is fully controllable because it enables the 'ant in the labyrinth' problem to be considered as the standard problem within the field of critical phenomena. The diversity of methods available in the theory of critical phenomena permits us to extend our treatment to

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the case of asymmetric hopping. In this paper we make the first very modest attempt to consider the case of asymmetric hopping within the RSRG scheme. Because very little is known from the available Monte Carlo data for this case (Pandey 1984, Stauffer 1985) more efforts are needed to clarify this situation. We hope that our work will stimulate more elaborate numerical and theoretical analysis in this field in the future.

Section 2 is devoted to the introductory analysis of symmetric and asymmetric random walk problems. Here we discuss the validity of Einstein's relation between the conductivity and diffusion and provide the definition of the small parameter in the case of applied external field. In § 3 we consider in some detail the field-theoretic formulation of the random asymmetric hopping problem. For the case of the pure lattice we reproduce the well known results (Lehr *et al* 1984) for the drift velocity and diffusion coefficient. We establish under what conditions recent results of Luck (1983), Fisher (1984) and Aronovitz and Nelson (1984) can be reproduced starting from the discrete microscopic model. We also briefly discuss the way of obtaining the weak disorder and the effective medium type (EMA) expansions on the basis of the functional integral type of approach. In § 4 we develop the RSRG scheme for the case of one dimension. Here we also establish a proper set of scaling variables and scaling functions needed for the multi-dimensional calculations. This then enables us to identify the conditions under which the single cluster phenomenological scaling results of Gefen *et al* (1983) can be obtained. Section 5 is devoted to simple illustrative two-dimensional calculations based on the rather crude Migdal approximation. We consider here only the fully directed case according to the terminology introduced by Redner (1982). For this case our results are as follows.

(a) The average root-mean-square displacement $\langle R(t) \rangle$ for the case of strong bias and the concentration of impurities slightly above the percolation threshold p^* behaves as $\langle R(t) \rangle \propto (p - p^*)^{\omega_1} t$ with $\omega_1 = \nu_p(1 + \theta)$, $\theta = (\mu - \beta_p)/\nu_p$, μ is the DC conductivity exponent, β_p and ν_p are known percolation exponents.

(b) For $p = p^*$ and times less than t_{cross} determined as $\min(\sigma^{-\nu_\sigma(2+\theta)}, D^{-\nu_p(2+\theta)})$, where σ is related to the strength of the bias and D to the magnitude of the macroscopic diffusion coefficient with ν_σ and ν_p being the critical exponents defined in the text, we obtained the standard result $\langle R(t) \rangle \propto t^{1/(2+\theta)}$ predicted by Gefen *et al* (1983). When $t \geq t_{cross}$ the system tends to crossover to the new regime not accessible by the present method.

In § 6 we reproduce the phenomenological results of Alexander and Orbach (1982) for the density of states which is valid for the frequencies greater than t_{cross}^{-1}. Section 7 is devoted to a brief discussion.

2. Symmetric and asymmetric random hopping. General results

The standard problem of random hopping on d-dimensional regular hyperlattice is described by equation of motion

$$\frac{\partial P_n}{\partial t} = \sum_m T_{nm} P_m(t) \tag{2.1}$$

with P_n the probability of being at site n at time t and T_{nm} is a hopping matrix which is assumed to be known. The first-order differential equation (2.1) is supplemented by the initial condition $P_n(0) = \delta_{no}$. In the absence of trapping, the hopping matrix

T_{nm} is subject to constraint

$$\sum_n T_{nm} = 0. \tag{2.2}$$

This constraint plays an important role in the subsequent analysis. For the case of symmetric hopping $T_{nm} = T_{mn}$. When $T_{nm} \neq T_{mn}$ we have the case of asymmetric hopping. This type of situation is usually realised when, for example, a particle having a charge is placed into a strong electric field (Lien and Shklovskii 1981). When the hopping matrix is random, the solution $P_n(t)$ must be properly averaged. The Laplace transform of equation (2.1) can be presented in the matrix form as

$$(s\boldsymbol{I} - \boldsymbol{H})\boldsymbol{P}(s) = \boldsymbol{I}_0 \tag{2.3}$$

where s is Laplace variable conjugate to t, $\boldsymbol{P}(s)$ is the Laplace transform of $\boldsymbol{P}(t)$, $\boldsymbol{I}$ is the unit matrix, $\boldsymbol{I}_0 = \delta_{no}$. The use of constraint (2.2) produces for the matrix $\boldsymbol{H}$ an expression:

$$H_{nm} = (1 - \delta_{nm}) T_{nm} - \delta_{nm} \sum_{m(\neq n)} T_{mn}. \tag{2.4}$$

The matrix H_{mn} is subject to the same type of constraint as T_{nm} in (2.2). In the random hopping problem the main quantity of interest is the averaged propagator

$$G(s) = \langle (s\boldsymbol{I} - \boldsymbol{H})^{-1} \rangle_0 = \langle \boldsymbol{K}^{-1} \rangle_0 \tag{2.5}$$

which gives $\langle P_n(t) \rangle_0$ as the inverse Laplace transform of $G_{no}(s)$. Here $\langle \cdots \rangle_0$ means some sort of impurity average. A knowledge of $\langle P_n(t) \rangle_0$ permits us to calculate moments according to

$$\langle R^{\delta}(t) \rangle_0 = \mathscr{L}^{-1} \left(\sum_n n^{\delta} \langle P_n(s) \rangle_0 \right) \tag{2.6}$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform and $\delta = 1, 2, 3, \ldots$. For the case of symmetric hopping only even moments are non-zero. In particular, knowledge of $\langle R^2(t) \rangle_0$ permits us to determine the diffusion coefficient D_{∞} via

$$D_{\infty} \propto \lim_{t \to \infty} \frac{\mathrm{d}}{\mathrm{d}t} \langle R^2(t) \rangle_0. \tag{2.7}$$

According to the theory of linear response the diffusion coefficient D_{∞} is connected to the DC conductivity σ_{DC} via Einstein's relation

$$\sigma_{\mathrm{DC}} = (e^2 n / T) D_{\infty}. \tag{2.8}$$

Here n is the concentration of carriers, e their charge, T is the temperature in energy units. If the applied field is strong enough, the theory of linear response is no longer applicable so that Einstein's relation (2.8) cannot be used. To see how this happens consider the expression for the current $j_{\mu} = ne\langle v_{\mu} \rangle_0$ where

$$\langle v_{\mu} \rangle_0 = \frac{\mathrm{d}}{\mathrm{d}t} \langle R_{\mu}(t) \rangle_0 \tag{2.9}$$

and μ is some component of the d-dimensional vector $\boldsymbol{R}(t)$: $(R_1(t), \ldots, R_{\mu}(t), \ldots, R_d(t))$. Using the equation of motion (2.1), we obtain

$$\langle v_{\mu} \rangle_0 = \sum_n n_{\mu} \left\langle \frac{\mathrm{d}P_n}{\mathrm{d}t} \right\rangle_0 = \sum_{n,m} (m_{\mu} - n_{\mu}) \langle T_{mn} P_n \rangle_0. \tag{2.10}$$

If now

$$T_{mn} = W_{mn} \exp[eE \cdot (m - n) \, e^{i\omega t} / T], \tag{2.11}$$

where $E \, e^{i\omega t}$ is the time-dependent uniform electric field directed parallel to the μth coordinate axis of the lattice, we then obtain, instead of (2.10), in the limit of small E the following expression

$$\langle v_\mu \rangle_0 = \left(\sum_{n,m} (m_\mu - n_\mu)(m_\mu - n_\mu)\langle W_{mn}P_n \rangle_0 \right) \frac{eE_\mu \, e^{i\omega t}}{T} \tag{2.12}$$

where we have taken into account the fact that W_{nm} is a symmetric matrix. Combining (2.12) with the expression for the current j_μ we reobtain Einstein's relation (2.8) with diffusion coefficient $D(t)$ defined as

$$D_{\mu\mu}(t) = \sum_{n,m} (m_\mu - n_\mu)(m_\mu - n_\mu)\langle W_{mn}P_n \rangle_0. \tag{2.13}$$

To see that such a defined diffusion coefficient is in agreement with (2.7) we have to consider

$$\frac{1}{2} \frac{d}{dt} \langle R_\mu^2(t) \rangle_0 = \langle R_\mu(t)v_\mu(t) \rangle_0$$

$$= \sum_n n_\mu n_\mu \left\langle \frac{dP_n}{dt} \right\rangle_0 = \sum_{nm} (m_\mu - n_\mu)^2 \langle W_{mn}P_n \rangle_0 \tag{2.14}$$

where the equation of motion (2.1) and the symmetry of the matrix W_{mn} were used. It is evident that (2.14) can be reduced to (2.13) if the symmetry of the matrix W_{mn} is taken into account. Hence indeed (2.13) (or (2.14)) is the desired diffusion coefficient. The given derivation permits us to determine the small parameter ε in the conduction–diffusion problem. Say we consider only the nearest-neighbour hopping then, evidently, the linear response approximation is valid if and only if

$$\varepsilon = \mathrm{Re}(eEl/T) \ll 1 \tag{2.15}$$

where l is the bond length. The above result is valid only in the single-electron approximation. Electron–electron interactions can substantially change the above picture (Gefen and Halley 1984). Simple restriction for the given bond (site) to be occupied by just one carrier in a given moment of time turns out to be not very important as recent Monte Carlo calculations indicate (Przyborowski and Woerkom 1985).

3. Field theoretic formulation of the random asymmetric hopping model

We begin here with a review of some well known results which we shall use in the subsequent sections. Use of the identity

$$Z(J) = \int \prod_i d\varphi_i \exp(-\tfrac{1}{2}\varphi A\varphi + \varphi J) = \pi^{m/2} (\det A)^{-1/2} \exp(\tfrac{1}{2}JA^{-1}J), \tag{3.1}$$

where m is the rank of the matrix A which supposedly must have its inverse, permits us to interpret the LHS of (3.1) as the generating function for obtaining all quantities

of interest. For the non-random case we can write

$$G_{mn}(s) = \frac{\partial}{\partial J_n} \frac{\partial}{\partial J_m} \ln Z|_{J=0}, \tag{3.2}$$

where instead of A we must now use the matrix K defined in (2.6). For the random case we have to compute the averaged generating function $\langle \ln Z(J) \rangle_0$ first in order to obtain $\langle G_{mn}(s) \rangle_0$ according to (3.2). Usually the replica trick method is used to obtain $\ln Z(J)_0$, i.e., we can write

$$\langle \ln Z(J) \rangle_0 = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} \langle (Z(J))^\alpha \rangle_0 \tag{3.3}$$

and the limit $\alpha \to 0$ is taken only at the very end of the calculations. Alternatively, we can use the real space method which is widely applied in the theory of dilute magnetism (Stinchcombe 1983). The asymmetric hopping problem turns out to be much more complex than the symmetric one as will be demonstrated below. Therefore both the replica trick and the real space method must be used depending upon the strength of the external field and/or degree of disorder. To understand better the essence of the problem we consider first of all the non-random one-dimensional case.

3.1. One-dimensional asymmetric hopping. Non-random case

Using formulae (3.1), (2.4) and (2.6) we have now to write down explicitly the expression for the quadratic form $\varphi K \varphi$. We have

$$\varphi K \varphi = \sum_i s \varphi_i^2 + \sum_i \varphi_i^2 \sum_{j(i \neq j)} T_{ji} - \sum_{\substack{i,j \\ (i \neq j)}} T_{ij} \varphi_i \varphi_j. \tag{3.4}$$

In the case of nearest-neighbour hopping we have

$$\varphi K \varphi = \sum_i s \varphi_i^2 + \sum_i \varphi_i^2 \sum_n T_{i+n,i} - \sum_i \sum_n T_{i,i+n} \varphi_i \varphi_{i+n} \tag{3.5}$$

where n is the unit d-dimensional vector. For the asymmetric case we have $T_{ij} \neq T_{ji}$. In the one-dimensional case we are left with the following matrix elements

$$\begin{aligned} T_{i,i+1} &= e^\varepsilon w, & T_{i+1,i} &= e^{-\varepsilon} w \\ T_{i,i-1} &= e^{-\varepsilon} w, & T_{i-1,i} &= e^\varepsilon w, \end{aligned} \tag{3.6}$$

where $\varepsilon = eEl/T$ (see (2.11) and (2.15)). Substitution of (3.6) into (3.5) produces the following result

$$\varphi K \varphi = \sum_i s \varphi_i^2 + 2w \sum_i \varphi_i^2 \cosh \varepsilon - w \sum_i \varphi_i (e^\varepsilon \varphi_{i+1} + e^{-\varepsilon} \varphi_{i-1}). \tag{3.7}$$

It is convenient to introduce Fourier transforms of the fields φ_j so that

$$\varphi_j = \text{constant} \times \sum_q e^{iqjl} \varphi_q. \tag{3.8}$$

The actual value of the constant in (3.8) is unimportant because fields can always be rescaled to absorb the constant. The use of (3.8) in (3.7) produces without delay

$$\sum_q \varphi_q \varphi_{-q} [s + 2w(\cosh \varepsilon - \cosh(iql + \varepsilon))]. \tag{3.9}$$

The expression in square brackets is just a well known expression for the inverse free propagator for the case of asymmetric hopping (Lehr *et al* 1984). To compute moments of the distribution function defined in (2.7) we can use the same methods as we have already used for the case of self-avoiding random walk (Kholodenko and Freed 1983), see also Lehr *et al* (1984). Using (3.8) we define the Fourier-Laplace transform of $P_n(t)$ as

$$P_q(s) = \mathcal{L}\left(\sum_n e^{-iqnl} P_n(t)\right), \tag{3.10}$$

then

$$\langle R^\delta(t)\rangle = \mathcal{L}^{-1}[(i)^\delta(\partial/\partial q^\delta)P_q(s)]_{q=0}. \tag{3.11}$$

Using (3.9)-(3.11) we obtain

$$\langle R(t)\rangle = (2wl\sinh\varepsilon)t \tag{3.12}$$

and from here, according to (2.9), we obtain

$$\langle v\rangle = 2wl\sinh\varepsilon. \tag{3.13}$$

The diffusion coefficient in the case of small ε can be obtained in the same way as has already been outlined in (2.13)-(2.15). In general, when ε is not small, we obtain instead

$$\langle R^2(t)\rangle = (2wl^2\cosh\varepsilon)t + \langle\sigma\rangle^2 t^2. \tag{3.14}$$

We can define following Lehr *et al* (1984) the diffusion coefficient $D(\varepsilon) = wl^2\cosh\varepsilon$. As we see in the case of arbitrary ε the use of formula (2.7) no longer produces the standard result for $D(\varepsilon)$. This result can be recovered only in the first order in small ε expansion. Consider now once again the expression (3.7), this time in the limit $\varepsilon = 0$. We now obtain

$$\varphi K\varphi = \sum_i s\varphi_i^2 + 2w\sum_i \varphi_i^2 - w\sum_i \varphi_i(\varphi_{i+1}+\varphi_{i-1})$$

$$= \sum_i s\varphi_i^2 + w\sum_i \varphi_i(\varphi_i-\varphi_{i+1}) + w\sum_i \varphi_i(\varphi_i-\varphi_{i-1}). \tag{3.15}$$

If now in the third sum we make the change $i \to i+1$, then we finally obtain the standard result

$$\varphi K\varphi = \sum_i s\varphi_i^2 + w\sum_i (\varphi_i-\varphi_{i+1})^2. \tag{3.16}$$

Now let $\varepsilon \neq 0$ but ε is small. Then using (3.7) we obtain, keeping terms up to first order in ε, the following result

$$\varphi K\varphi = \sum_i s\varphi_i^2 + w\sum_i (\varphi_i-\varphi_{i+1})^2 + \varepsilon w\sum_i \varphi_i(\varphi_{i-1}-\varphi_{i+1}). \tag{3.17}$$

Using Fourier transform representation (3.8) for φ_i and formula (3.11) we recover the result of linear response theory, namely $\langle v\rangle \propto \varepsilon$, $D = wl^2$. In the case if w is a random variable the situation is not so trivial even when $\varepsilon \ll 1$. Therefore we would now like to consider this situation in some detail.

3.2. Many-dimensional asymmetric hopping. Random case ($\varepsilon \ll 1$)

Formula (3.17) can be easily extended to arbitrary dimensions. We want, however,

not only to extend it but to make some connections with other works. To do so we notice first that, by construction, when w is a random variable, we obtain the situation when both the diffusion coefficient and the drift force are random variables. Using (2.11) and (3.5) we now obtain

$$\varphi K \varphi = \sum_i s\varphi_i^2 + \sum_i \varphi_i^2 \sum_n w_{i+n,i} \exp\left(\frac{eEn}{T}\right) - \sum_i \sum_n w_{i,i+n} \exp\left(\frac{eEn}{T}\right) \varphi_i \varphi_{i+n}. \tag{3.18}$$

When the parameter ε defined in (2.16) is less than unity, we can rewrite (3.18) (absorbing the unimportant constants)

$$(3.18) = \sum_i s\varphi_i^2 + \sum_i \varphi_i^2 \sum_n w_{i+n,i} - \sum_i \sum_n w_{i,i+n} \varphi_i \varphi_{i+n}$$

$$+ \sum_i \sum_n w_{i+n,i} En \varphi_i (\varphi_i - \varphi_{i+n}) + O(\varepsilon^2) \tag{3.19}$$

where it was taken into account that the matrix w_{ij} is symmetric. When the matrix w_{ij} is random, we have to consider the following averaged replicated functional integral

$$\langle Z(j)\rangle_0 = \left\langle \int \prod_i d\varphi_i \exp\left(-\tfrac{1}{2}\sum_i s\varphi_i^2 - \tfrac{1}{2}\sum_i \sum_n w_{i,i+n}(\varphi_i - \varphi_{i+n})^2\right) \right.$$

$$\left. \times \exp\left(-\sum_i \sum_n w_{i+n,i} \frac{En}{2}\varphi_i(\varphi_{i-n} - \varphi_{i+n}) + \sum_i \varphi_i j_i\right) \right\rangle_0 \tag{3.20}$$

where $\varphi_i = \prod_{\alpha=1}^n \varphi_{\alpha i}$ and the vector n is directed now only along the positive semi-axes of the d-dimensional hypercubic lattice. Clearly, in one dimension the quadratic form in the exponent of (3.20) coincides with that of (3.17) as it must. Let now $w_{i,i+n} = w_0 + \delta w_{i,i+n}$ where w_0 is non-random and δw is a random part of the matrix w_{ij}. Such decomposition can be realised, for example, if we expand around the effective medium (EMA) results. Let, for instance, $\delta w_{i,i+n}$ be a Gaussian random variable, i.e., we assume that

$$\langle \delta w_{i,i+n} \delta w_{j,j+m}\rangle_0 = \Delta \delta_{ij} \delta_{nm}. \tag{3.21}$$

Then we obtain using (3.20) and (3.21) the following result

$$\langle Z(j)\rangle_0 = \int \prod_i d\varphi_i \exp\left(-\tfrac{1}{2}\sum_i s\varphi_i^2 - \tfrac{1}{2}\sum_i \sum_n w_0(\varphi_i - \varphi_{i+n})^2\right)$$

$$\times \exp\left(-\sum_i \sum_n w_0 \frac{E \cdot n}{2}\varphi_i(\varphi_{i-n} - \varphi_{i+n})\right)$$

$$\times \exp\left(+\tfrac{1}{8}\Delta \sum_i \sum_n [E \cdot n \varphi_i(\varphi_{i-n} - \varphi_{i+n})]^2 + \ldots + \sum_i \varphi_i j_i\right) \tag{3.22}$$

where the dots in the exponent denote, in general, higher order averages which are irrelevant for the renormalisation group treatment. Expression (3.22) should be compared with the starting expressions of Luck (1983) and Fisher (1984) who subsequently analyse it using the renormalisation group (RG) method. This is achieved by going to the continuum limit in (3.22) and absorbing all the unimportant constants by an

appropriate rescaling of fields. We obtain then

$$\langle Z(j)\rangle_0 = \int D[\varphi] \exp\left(-\tfrac{1}{2} \int d^d x[s\varphi^2 + w_0(\nabla\varphi)^2]\right)$$

$$\times \exp\left(-\tfrac{1}{2} \int d^d x \, \varphi V_\mu^0 \nabla_\mu \varphi\right)$$

$$\times \exp\left(\frac{g_0}{2} \int d^d x(\varphi V_\mu^0 \nabla_\mu \varphi)^2 + \int d^d x \, \varphi \cdot j\right), \qquad (3.23)$$

here $\mu = 1 - d$, summation over repeated indices is assumed, $V_\mu^0 = w_0 E_\mu$, and the actual value of the coupling constant g_0 is unimportant. The functional integral (3.23) is converted to that *postulated* by Luck (1983) if the following simplification is made:

$$\int d^d x(\varphi V_\mu^0 \nabla_\mu \varphi)^2 \simeq V_0^2 \int d^d x(\varphi \nabla_\mu \varphi)^2. \qquad (3.24)$$

This simplification is justifiable if instead of a monocrystal with impurities we consider an amorphous solid. Then in addition to the impurity average we also have to perform an angular average (Landau and Lifshitz 1954). This then would eliminate the exponential term linear in E and leave only the term quadratic in E decoupled according to (3.24). Such a model then coincides with that proposed by Fisher (1984) and is in disagreement with that proposed by Luck (1983). Note that his complex n-component scalar field can always be converted to the $2n$-component real field and because the limit $n \to 0$ must be taken anyway these two are identical models. Fisher's model by no means exhausts the class of models appropriate for the $\varepsilon \ll 1$ regime. For instance, if impurities are correlated we may have instead of relation (3.21) the following more general relation

$$\langle \delta w_{i,i+n} \delta w_{j,j+m}\rangle_0 = f(li - jl)\delta_{nm} \qquad (3.25)$$

where $f(|x|)$ is some known function. Alternatively, we can have instead of (3.25) the following correlator

$$\langle \delta w_{i,i+n} \delta w_{j,j+m}\rangle_0 = \begin{cases} \Delta_\| \delta_{ij} & \text{if } n \parallel m \\ \Delta_\perp \delta_{ij} & \text{if } n \perp m. \end{cases} \qquad (3.26)$$

Such a type of model is close to that recently proposed by Aronovitz and Nelson (1984). The RG treatment of the last model shows some new features not available from the anaysis of the Fisher–Luck model. This indicates that critical behaviour is disorder-specific in general. More complicated situations are conceivable as well. Given analysis shows under what conditions the theory of linear response briefly described in §2 is applicable. We see that for the amorphous solids, even when $\varepsilon \ll 1$, the theory of linear response breaks down because the term linear in E drops out due to angular averaging. For the case of monocrystals with impurities the term in the exponent of (3.22) which is proportional to E^2 should be considered as a small perturbation, by construction, so that the theory of linear response is applicable.

We have mentioned that the decomposition of the matrix w_{ij} onto fluctuating and non-fluctuating parts can be associated with the expansion around the EMA result. Now we would like to consider this situation in more detail.

3.3. Weak disorder and EMA expansions

Recently several authors have considered the weak disorder expansions for the asymmetric hopping problem. In one dimension results are much more impressive than in higher dimensionalities. In particular, Bernasconi and Schneider (1983) gave an exact asymptotic result for $R(t)$ for the case of percolative-like distribution for the hopping rates. More specifically, they considered a disordered chain for which with probability p the hopping rates are

$$T_{n+1,n} = u \qquad T_{n,n+1} = 0$$

and with probability $1 - p$ the hopping rates are

$$T_{n+1,n} = \lambda\sigma, \qquad T_{n,n+1} = \sigma,$$

where λ, u, σ are some non-negative numerical constants. Khantha and Balakrishnan (1984) also considered hopping conductivity in a one-dimensional bond-percolation model in a constant external field. They failed to produce an exact result as they encountered serious mathematical difficulties. They provide only an asymptotic result for the case of low and high frequencies. Another more general type of probability distribution was considered by Derrida and Orbach (1983) who also provide low and high frequency expansions and expansions in terms of weak disorder. Similar results were obtained then by Biller (1984) and Lehr *et al* (1984). Many-dimensional generalisations of results of Derrida and Orbach were made by Derrida and Luck (1983) which is much less complete in terms of quality and quantity of obtained results compared with the one-dimensional case. Finally Stephen (1981) was the first who considered the case of asymmetric hopping within the EMA approximation using the functional integral method. Very simple derivation of the EMA approximation with help of the functional integral method but without the replica trick was given in the paper by Kholodenko and Freed (1984). It is instructive, for completeness, to discuss here both weak disorder and EMA approximations in order to see their connection with results of § 3.2. We can write for the matrix elements $w_{i+n,i} = T_{i+n,i} + (w_{i+n,i} - T_{i+n,i}) = T_{i+n,i} + \Delta T_{i+n,i}$ and similarly $w_{i,i+n} = T_{i,i+n} + (w_{i,i+n} - T_{i,i+n}) = T_{i,i+n} + \Delta T_{i,i+n}$ where T_{ij} is a non-random matrix. The difference between the EMA and weak disorder expansion lies in the fact that in the first case the matrix T_{ij} is found self-consistently (Kholodenko and Freed 1984) whereas in the second case the matrix T_{ij} is assumed to be known (Lehr *et al* 1984). For the weak disorder case we have to substitute the above matrix elements into (3.18) and then in the functional integral (3.1) and (3.3). Instead of considering the partition function (3.3) it is instructive to consider from the beginning the averaged propagator (the non-averaged propagator was defined in (3.2)). Then expanding the exponents containing ΔT_{ij} matrix elements into power series, performing a trivial Gaussian integration with the use of Wick's theorem (Itzykson and Zuber 1980) and subsequent impurity averaging produces the desired weak disorder expansion. Alternatively, we could perform the impurity average first and then accomplish the functional integration order by order in powers of averaged ΔT_{ij} using known free Gaussian propagators. The last option then would lead directly to results discussed in § 3.2. The EMA expansion, in principle, can be treated in exactly the same way with the two available options described above. Because the renormalisation group analysis is not too sensitive with respect to the actual numerical values of the bare coupling constants, for the purposes of this analysis it is sufficient to keep only terms quartic in fields φ_i as was demonstrated in § 3.2. When

the non-critical regime is considered this is no longer true and the detailed analysis is necessary. Because we are interested here only in the development of the renormalisation group method, we omit all technical details associated with actual computations outside the critical regime. Before we finish this subsection, we have to notice as well some differences between the results of § 3.3 compared with that of § 3.2. In § 3.3 we do not require the parameter ε to be small. Instead, we require only that the disorder must be weak. Because of that there is a potential possibility of variety of critical regimes which was already discovered at the EMA level by Stephen (1981). We shall not go into these details here. We will be interested in this work in the case of arbitrary percolative-like disorder and strong electric field. We have to note, though, that in high enough fields the approximation of nearest-neighbour jump rates in the master equation (2.1) may no longer be valid. The approach which we shall use below does not take such effects into account.

4. Real space renormalisation of the random asymmetric hopping model ($\varepsilon > 1$). One-dimensional case

Recently Lien and Shklovskii (1981) provided a qualitative picture of random hopping conduction in a strong electric field. They correctly concluded that for DC conduction the above hopping problem is effectively reduced to the problem of directed percolation (Obukhov 1980). More accurately, we may say that the random resistors network problem and ordinary percolation (for $\varepsilon \ll 1$) are related to each other in the same way as the random diode problem is related to the directed percolation (for $\varepsilon \gg 1$) (Redner 1982, Harms and Straley 1982). In practice, the situation is not so simple, and we have to distinguish between the case of the perfect lattice with impurities and that of the amorphous lattice when, in addition to the impurity average, the angular averaging must be performed. In the last case the above analogy is evidently lost and the situation is much more complex. When the time-dependent problems are considered the above analogy with directed percolation is of little use in the very same way as for the case of ordinary percolation and time-dependent diffusion–conduction.

Recently a couple of works have appeared in which an attempt had been made to describe the time-dependent asymmetric random hopping. Some of these attempts have already been listed in § 3. Here we provide the additional references. Monte Carlo simulations of biased random walk performed by Pandey (1984) indicate rather complicated interplay between the strength of the biased field, the extent of the disorder and the classical diffusion against drift behaviour. Pandey concludes that for the mean displacement $\langle R(t) \rangle$ the appropriate formula is

$$R(t) \sim t^k \tag{4.1}$$

where k is some time-dependent exponent which is in general unknown. A somewhat different conclusion was arrived at by Dhar (1984). He postulates the same dependence for $\langle R(t) \rangle$ as in (4.1) but with k defined as $k = (\xi_p \ln e^\varepsilon)^{-1}$ where we used our notation (see definition of ε in (2.15)) and ξ_p being the percolation correlation length. The above biased hopping takes place on the bond (site) percolative lattice near the percolation threshold. Theoretical calculations of White and Barma (1984) were made for the case of a Bethe lattice. They are in qualitative agreement with Dhar's hypothesis. Unlike the case of unbiased diffusion, where a large body of results have been

accumulated, the case of biased diffusion does not enjoy such completeness. The most recent Monte Carlo results of Stauffer (1985) are in qualitative agreement with Dhar's conjecture. Here we would like to develop a RSRG method in order to find out under what conditions result (4.1) can be obtained.

4.1. Real space renormalisation for the case of the one-dimensional non-random symmetric and asymmetric hopping

In § 3 we have already obtained all the results needed for the renormalisation group analysis. For convenience, we reproduce them here once again in a more systematic way. Using (3.1) and (3.7) we now obtain

$$Z(j) = \int \prod_i d\varphi_i \exp\left(-\sum_i [s + w \cosh \varepsilon]\varphi_i^2 - \frac{w}{2}\sum_i \varphi_i(e^\varepsilon \varphi_{i+1} + e^{-\varepsilon}\varphi_{i-1} + \sum_i \varphi_i j_i\right). \tag{4.2}$$

To describe the effective macroscopic conductivity (diffusion) there is actually no need to calculate the averaged propagator $\langle G_{nm}^{(s)} \rangle_0$, defined in (3.2), than to compute $\langle R(t) \rangle$, $(d/dt)\langle R(t) \rangle_0$, etc. For the case $\varepsilon = 0$, due to the Gaussian nature of the 'kinetic' term in (3.16) the effective diffusion coefficient will naturally emerge as some multiplier in front of the 'kinetic' term $\frac{1}{2}(\varphi_i - \varphi_{i+1})^2$ (see (3.16)) during the course of real space renormalisation. When $\varepsilon \neq 0$, the situation is not so simple which was already demonstrated in (3.14). If, however, following Derrida and Luck (1983), we define the diffusion tensor D for d-dimensional hyperlattice as

$$D_{\mu\nu} = \frac{1}{2} \lim_{t \to \infty} \frac{d}{dt}(\langle R_\mu(t)R_\nu(t)\rangle_0 - \langle R_\mu(t)\rangle_0 \langle R_\nu(t)\rangle_0), \tag{4.3}$$

then using this definition in one dimension and combining it with (3.14) produces

$$D = wl^2 \cosh \varepsilon. \tag{4.4}$$

Using result (4.4) together with (2.10), (3.12), (3.14) and assuming that $\varepsilon \gg 1$ we can rearrange (4.2) as follows ($j = 0$):

$$Z(w, \varepsilon, s) = \int \prod_i d\varphi_i \exp\left(-\sum_i s\varphi_i^2 - \frac{w\,e^\varepsilon}{2}\sum_i (\varphi_i - \varphi_{i+1})^2\right)$$

$$\times \exp\left(-\frac{w\,e^\varepsilon}{2}\sum_i \varphi_{i+1}(\varphi_i - \varphi_{i+1})\right). \tag{4.5}$$

The continuum version of (4.5) reproduces evidently results for D and $\langle v \rangle$ given by (3.13) and (4.4) for the case of large ε. From here we see that indeed the coefficient in front of the 'kinetic' term in (4.5) can be associated with D and the coefficient in front of the 'gradient' term is just the average velocity. Such identification removes the necessity to consider the partition function Z for the case of arbitrary current J.

After these preliminaries we need to rewrite (4.5) in the following form:

$$Z(w, \varepsilon, s) = \int \prod_i d\varphi_i A_{w,\varepsilon,s}(i, i+1) \tag{4.6}$$

where

$$A_{w,\varepsilon,s}(i, i+1) = \exp[-\tfrac{1}{2}w\, e^\varepsilon(\varphi_i - \varphi_{i+1})^2 - \tfrac{1}{2}w\, e^\varepsilon \varphi_{i+1}(\varphi_i - \varphi_{i+1})]\exp(-\tfrac{1}{2}s\varphi_i^2 - \tfrac{1}{2}s\varphi_{i+1}^2).$$
(4.7)

Even though at this time D and v coefficients are looking the same, it is necessary to keep track of each of them separately in view of future generalisations. Therefore it is convenient to rewrite (4.7) as follows:

$$A_{D,v,s}(i, i+1) \exp[-\tfrac{1}{2}D(\varphi_i - \varphi_{i+1})^2 - \tfrac{1}{2}v\varphi_{i+1}(\varphi_i - \varphi_{i+1})]\exp[-\tfrac{1}{2}s\varphi_i^2 - \tfrac{1}{2}s\varphi_{i+1}^2)$$
(4.7a)

with obvious definitions of D and v.

Consider now the following integral

$$I = \int d\varphi_2\, A_{D,v,s}(1, 2)A_{D,v,s}(2, 3).$$
(4.8)

Straightforward calculation of (4.8) then produces

$$I = \left(\frac{2\pi}{2D + 2s + v}\right)^{1/2} \exp[-\tfrac{1}{2}D'(\varphi_1 - \varphi_3)^2 - \tfrac{1}{2}v'\varphi_3(\varphi_1 - \varphi_3)]$$
$$\times \exp[-\tfrac{1}{2}s'(l)\varphi_1^2 - \tfrac{1}{2}s'(r)\varphi_3^2],$$
(4.9)

where

$$D' = \tfrac{1}{4}[(2D - v)^2/(2D + 2s - v)] + \tfrac{1}{2}v$$
(4.10)

$$v' = v,$$
(4.11)

$$s'(1) = s'(r) = s' = s + D - \tfrac{1}{2}v - \tfrac{1}{2}[(2D - v)^2/(2D + 2s - v)].$$
(4.12)

In order to analyse recurrences (4.10)-(4.12) it is useful first to formally put $b = 0$. This is perfectly permissible because, in view of (4.14), v plays just a role of parameter. When $v = 0$ we obtain

$$D' = D^2/2(D + s),$$
(4.13)

$$s' = s + sD/(s + D).$$
(4.14)

Recurrences (4.16) and (4.17) have the following sets of fixed points

$$s^* = 0, \qquad D^* = 0,$$
(4.15a)

$$s^* = 0, \qquad D^* = \infty.$$
(4.15b)

When v is non-zero the first set (4.15a) can no longer be considered, hence we do not consider it below at all. If we are interested in the long time limit of the hopping process $t \to \infty$ ($s \to 0$), we need to check first whether the value $s = s^* = 0$ leads to physically acceptable results for recurrences (4.10)-(4.12). For $s^* = 0$ we obtain $D^* = \tfrac{1}{2}v$ which is clearly different from the values given by (4.15a) and (4.15b). Hence we see that strong bias may change the physical picture in one dimension provided that operator D is relevant at this fixed point. To find this, we would like to outline first the basic scaling relations for the case of non-random hopping. First we assume that $\langle R^\delta \rangle$, given by (2.6), can be written as

$$\langle R^\delta \rangle = l^\delta f(D, v, t)$$
(4.16)

where the generally unknown function f is assumed to be dimensionless. This can be achieved by introduction of some microscopic time scale τ_0 such that in fact $f(D, v, t) = f(\tau_0 D, \tau_0 v, \tau_0^{-1} t)$. We choose $\tau_0 = 1$ subsequently unless otherwise specified. Alternatively, (4.16) can be rewritten, for example, as

$$\langle R^\delta \rangle = l^\delta f(Dt, vt) \tag{4.16a}$$

etc, so that introduction of τ_0 becomes unnecessary. For the statistical mechanics treatment the first option is preferable. The introduction of τ_0 comes from the fact that the equation of motion (2.1) is homogeneous. In the field-theoretic renormalisation (Itzykson and Zuber 1980) arbitrary τ_0 corresponds to the arbitrary length scale parameter for the renormalised quantities. Following De Gennes (1979) we write

$$l'^\delta f(D', v', t') = l^\delta f(D, v, t) \tag{4.17}$$

to be contrasted with Ohtsuki and Keyes (1984a, b). Notice the correct choice of relations like (4.17) is highly non-trivial and, in general, affects the final results. When $v = 0$ the exponent δ can only have values $2, 4, \ldots$. In this case consider, for example, $\delta = 2$. We have

$$\langle R^2 \rangle = l'^2 f(D', t')$$
$$= l'^2 b^2 f(Db^{1/\nu_D}, tb^{-1/\nu_s}) \tag{4.18}$$

where we formally introduced the exponents ν_D and ν_s via relations $D'/D = b^{1/\nu_D}$, $t'/t = b^{-1/\nu_s}$ with b being a decimation parameter (in our case $b = 2$). Using definition (2.7) we also obtain

$$D(t) = l^2 b^{2-1/\nu_s} \tau_0^{-1} \tilde{f}(Db^{1/\nu_D}, tb^{-1/\nu_s}) \tag{4.19}$$

where we reintroduced τ_0 for clarity. If D in (4.13) can be interpreted as a microscopic conductivity, then the recurrence (4.13) is nothing but the Ohm's law for two sequentially joined conductors for $s = 0$. This observation permits us to develop the scaling theory of conductivity in the linear response regime given by (2.12) and (2.13). According to Lubensky (1979) we can write for the averaged conductivity $\langle v \rangle$ before and after decimation the following relation

$$\langle \sigma \rangle / D = (\langle \sigma' \rangle / D') b^{d-2}. \tag{4.20}$$

Taking into account (4.13) and (4.14) this can be equivalently presented as

$$\langle \sigma \rangle / D = b^{d-2} f(D' b^{-1/\nu_D}, s' b^{-1/\nu_s}). \tag{4.21}$$

When $v = 0$, the situation is more complicated. Consider, for example, the case $\delta = 1$ in (4.17),

$$\langle R \rangle = l' f(D', v', t')$$
$$= lb f(Db^{1/\nu_D}, vb^{1/\nu_v}, tb^{-1/\nu_s}) \tag{4.22}$$

where we have introduced the new exponent ν_v via relation $v'/v = b^{1/\nu_v}$. When $v \to 0$ we should expect $\langle R \rangle \to 0$. Therefore (4.22) in this limit should be reduced to

$$\langle R \rangle = lb^{1+1/\nu_v} v \tilde{f}(Db^{1/\nu_D}, tb^{-1/\nu_s}) + O(v^2). \tag{4.23}$$

It can be shown (Nelson and Fisher 1975) that the formal relations like $D'/D = b^{1/\nu_D}$ are valid only in the vicinity of the fixed points, if they exist, of the corresponding recurrences. Therefore, we actually have to write instead of $D'/D = b^{1/\nu_D}$ the following equation $(D' - D^*)/(D - D^*) \equiv \delta D'/\delta D \simeq dD'/dD|_{D^*,s^*} = \lambda_D = b^{1/\nu_D}$ where $D^*, s^*, \ldots$ are the corresponding fixed points for D, s, etc. The same must also be true for all other operators. Taking into account these comments, the exponents ν_D, ν_s are defined in the usual way as

$$\nu_0 = \ln b / \ln \lambda_0 \tag{4.24}$$

where the subscript 0 stands for D, s, etc. Using the recurrence (4.10) we obtain without delay $\lambda_D = \frac{1}{2}$ and, hence, the operator D is irrelevant at $D^* = \frac{1}{2}v$. The only relevant fixed point for D is given by (4.15b). For this point it is convenient to introduce the inverse quantity $\rho = D^{-1}$ so that $-\nu_D = \nu_\rho$. Using (4.10) and (4.13) we obtain $\nu_\rho = 1$. The situation with λ_s is somewhat delicate. Going back to the original expressions (4.5) and (4.7) we notice that initially in (4.5) we had just one site term associated with the s variable. By going from (4.5) to (4.7) we created two site terms associated with s for each bond. This procedure is somewhat artificial and was introduced here by analogy with similar situations, e.g. one-dimensional Ising model in the magnetic field (Stanley 1971). In order to compute the critical exponent we have to take into account both $s'(1)$ and $s'(r)$ recurrences. This is especially true for the disordered case when $s'(1)$ and $s'(r)$ are not necessarily the same (see equations (4.35), (4.39)). Using the above arguments, we have to write instead of (4.24)

$$z = \nu_s^{-1} = [\ln(\lambda_s^{(l)} + \lambda_s^{(r)})]/\ln b \tag{4.25}$$

where $\lambda_s^{(l,r)} = ds'(l, r)/ds|_{s^*,D^*}$. Using s recurrences (4.12) and (4.14) together with (4.25) we obtain $z = 2$. Going back to the expression (4.18) for $\langle R^2 \rangle$ and requiring $tb^{-1/\nu_s} = 1$ we obtain

$$\langle R^2 \rangle = l^2 t^{2\nu_s} \varphi(\delta \rho t^{\nu_s/\nu_\rho}). \tag{4.26}$$

As $\langle R^2 \rangle \propto t$ we expect that the scaling function behaves like $\varphi(x) \sim x^{\omega_D}$ for large x. This gives the following value for ω_D

$$\omega_D = \nu_\rho (1/\nu_s - 2). \tag{4.27}$$

Using previously obtained values of ν_s and ν_ρ we find $\omega_D = 0$ and $\langle R^2 \rangle \propto t$, as anticipated. Requiring $tb^{-1/\nu_s} = 1$ in (4.19) and repeating the same arguments produces the standard result $D = \text{constant}$. The situation with the conductivity will be considered later, when the disorder is introduced. When the bias is present, use of (4.22) produces

$$\langle R \rangle = lt^{\nu_s} \varphi(\delta \rho t^{\nu_s/\nu_\rho}, \delta v t^{\nu_s/\nu_v}). \tag{4.28}$$

In one dimension the situation is simple because of the relation (4.11). This gives $\nu_v^{-1} = 0$ so that the scaling function $\varphi(x)$ in (4.28) should behave like $\varphi(x) \sim x^{\omega_v}$ for large x, to reproduce the result $\langle R \rangle \propto t$. The given analysis completes the treatment of the one-dimensional non-random case.

4.2. Real space renormalisation group for the case of one-dimensional random symmetric and asymmetric hopping

In the previous subsection we have demonstrated how the RG method can produce results which are in qualitative agreement with exact calculations. The simplicity of

the RG method permits us to extend it to the case of bond disorder described by the percolative like jump probability which for the one-dimensional case can be written as

$$W(w_{i,i+1}) = p\delta(w_{i,i+1} - w^0) + (1 - P)\delta(w_{i,i+1}). \tag{4.29}$$

Using this probability distribution we can write for the 'free energy' per bond the following expression

$$\langle f \rangle_0 = \lim_{N \to \infty} \frac{1}{N_B} \int \prod_i dw_{i,i+1} W(w_{i,i+1}) \ln Z[w_{i,i+1}] \tag{4.30}$$

where $N_B = \frac{1}{2}\chi N$ with χ being a coordination number of the lattice which in one-dimensional case is just 2. The replicated version of this model for the case of zero bias was previously considered by Stephen and Kariotis (1982). In (4.30) we omitted unimportant for the present indices ε and s in the partition sum defined in (4.6) and (4.7) with jump probabilities w now being random variables. We shall call the model defined by (4.6), (4.7), (4.28) and (4.30) as diluted modified Gaussian model (DMGM) to be contrasted with the diluted Gaussian model (DGM) for $\sigma = 0$. To develop the RG method for DMGM we have to consider instead of integral (4.8) the following generalisation

$$I = \int d\varphi_2 \, A_{D_1,v_1,s}(1,2) A_{D_2,v_2,s}(2,3) \tag{4.31}$$

where now pairs (D_1, v_1) and (D_2, v_2) can generally be different. It is instructive first to consider the case $v_1 = v_2 = 0$, $s = 0$. The straightforward calculation then produces

$$I_{\sigma=0,s=0} = \left(\frac{2\pi}{D_1 + D_2}\right)^{1/2} \exp\left(-\frac{D_1 D_2}{2(D_1 + D_2)}(\varphi_2 - \varphi_3)^2\right) \tag{4.32}$$

and we can again immediately recognise Ohm's law if instead of D we would have conductivities σ_1 and σ_2, i.e. $\sigma^T = \sigma_1\sigma_2/(\sigma_1 + \sigma_2)$ is just the total conductivity of two sequentially joined conductors. If $v_1 = v_2 = 0$ but $s \neq 0$ we obtain DGM where $(v \rightleftarrows D)$

$$\sigma_{12} = \frac{\sigma_1\sigma_2}{2[s + \frac{1}{2}(\sigma_1 + \sigma_2)]}, \tag{4.33}$$

$$s'(l) = \frac{[s + \frac{1}{2}(3\sigma_1 + \sigma_2)]}{[s + \frac{1}{2}(\sigma_1 + \sigma_2)]}, \tag{4.34}$$

$$s'(r) = s\frac{[s + \frac{1}{2}(\sigma_1 + 3\sigma_2)]}{[s + \frac{1}{2}(\sigma_1 + \sigma_2)]}. \tag{4.35}$$

Here we already observe the differences between the left and the right recurrences as it was explained before equation (4.25). Notice that prescription (4.25) restores the symmetry with respect to the permutation $\sigma_1 \rightleftarrows \sigma_2$. In the most general case we obtain

$$I = \left(\frac{2\pi}{D_1 + D_2 + 2s - v_1}\right)^{1/2} \exp[-\frac{1}{2}D'(\varphi_1 - \varphi_3)^2 - \frac{1}{2}v'\varphi_3(\varphi_1 - \varphi_3)]$$

$$\times \exp[-\frac{1}{2}s'(l)\varphi_1^2 - \frac{1}{2}s'(r)\varphi_2^2) \tag{4.36}$$

where

$$D' = \frac{1}{2}v_2 + \frac{1}{4}\frac{(2D_2 - v_2)^2}{(D_1 + D_2 + 2s - v_1)}, \tag{4.37}$$

$$v' = v_2, \tag{4.38}$$

$$s'(l) = s + D_1 - \tfrac{1}{2}v_2 - \frac{1}{4}\frac{[(2D_2 - v_2)^2 + (2D_1 - v_1)^2]}{[D_1 + D_2 + 2s - v_1]}, \tag{4.39a}$$

$$s'(r) = s + D_2 - \tfrac{1}{2}v_2 - \frac{1}{2}\frac{(2D_2 - v_2)^2}{(D_1 + D_2 + 2s - v_1)}. \tag{4.39b}$$

Given recurrences reduce to those previously obtained, see (4.10)-(4.12), when $D_1 = D_2 = D$, $v_1 = v_2 = v$. Unlike the symmetric case, equation (4.33), these recurrences are explicitly asymmetric with respect to permutation $D_1 \rightleftarrows D_2$. We have noticed already in the previous subsection another difference between the symmetric and asymmetric hopping which lies in the fact that even in the pure non-random case the sets of fixed points are different, in general, for symmetric and asymmetric hopping. After one decimation the probability distribution (4.29) must be adjusted accordingly (Stinchcombe 1983). Unlike the symmetric case, where the new probability distribution can be presented in the form

$$W'(w'_{i,i+1}) = (1-p')\delta(w'_{i,i+1}) + p'\delta(w'_{i,i+1} - f_{i,i+1}(w,p) \tag{4.40}$$

with known recurrence function $f(w, p)$, now we need to slightly modify the rules in order to obtain the desired result. To do so, introduce instead of (4.30) the following probability distributions (index i is omitted in the following)

$$W_D(D) = p\delta(D - D_0) + (1-p)\delta(D), \tag{4.41}$$

$$W_v(v) = p\delta(v - v_0) + (1-p)\delta(v) \tag{4.42}$$

where D_0, v_0 are some initial values of D and v respectively. Then, after one decimation, we obtain

$$W'_D(D') = p'\delta(D' - f_D(D, v, p, s)) + (1-p')\delta(D'), \tag{4.43}$$

$$W_v(v') = p'\delta(v' - f_v(D, v, p, s)) + (1-p)\delta(v'). \tag{4.44}$$

From the symmetric hopping case (Stinchcombe 1983) we know that

$$p' = p^2 \tag{4.45}$$

and functions f_D and f_v can be obtained from the following relations

$$p'D' = p^2\left(\frac{1}{4}\frac{(2D - v)^2}{(2D + 2s - v)} + \frac{v}{2}\right) + p(1-p)\left(\frac{(2D - v)^2}{4(D + 2s)} + \frac{v}{2}\right), \tag{4.46}$$

$$p'v' = pv. \tag{4.47}$$

Relations (4.46) and (4.47) must be augmented by the appropriate relation for the s recurrence. Following ideas of Jayaprakash et al (1978), in order to obtain the corresponding relation for s we have to demand that when D and v are both zero the renormalisation should leave the variable s at the undecimated sites unchanged. This then produces, in view of (4.25), the following result

$$\frac{s'}{2} = p^2\left(s + D - \frac{v}{2} - \frac{1}{2}\frac{(2D - v)^2}{(2D + 2s - v)}\right) + p(1-p)\left(s + \frac{D}{2} - \frac{v}{2} - \frac{3}{8}\frac{(2D - v)^2}{(D + 2s)}\right)$$

$$+ p(1-p)\left(s + \frac{D}{2} - \frac{1}{8}\frac{(2D - v)^2}{(D + 2s - v)}\right) + (1-p)^2 s. \tag{4.48}$$

Formulae (4.45)-(4.48) accomplish the task of obtaining the recurrences for the one-dimensional random asymmetric hopping and permit us to develop the renormalisation group analysis. As in the non-random case, we begin with analysis of scaling functions. Introducing the new scaling exponent ν_p in the usual way, according to (4.24), we can write now the following scaling relations

$$\langle R^2 \rangle = l^2 b^2 f(\delta\rho b^{1/\nu_p}, tb^{-1/\nu_s}, \delta p b^{1/\nu_p}), \qquad v = 0, \tag{4.49}$$

$$D(t) = l^2 \tau_0^{-1} b^{2-1/\nu_s} f(\delta\rho b^{1/\nu_p}, tb^{-1/\nu_s}, \delta p b^{1/\nu_p}), \qquad v = 0, \tag{4.50}$$

$$\langle R \rangle = lbf(\delta\rho b^{1/\nu_p}, \delta v b^{1/\nu_v}, tb^{-1/\nu_s}, \delta p b^{1/\nu_p}), \tag{4.51}$$

$$\frac{\langle \sigma \rangle}{D} = b^{d-2} f(\delta\rho' b^{-1/\nu_p}, \delta s' b^{-1/\nu_s}, \delta p' b^{-1/\nu_p}), \qquad v = 0. \tag{4.52}$$

Now consider the last relation. Let $\delta s' = 0$, so that we have the case of random resistors network. Demanding that $\delta p' b^{-1/\nu_p} = 1$ we obtain

$$\langle \sigma \rangle / D = (\delta p')^{\nu_p(d-2)} \varphi(\delta\rho'(\delta p)^{-\zeta}) \tag{4.53}$$

where the crossover exponent ζ is defined as $\zeta = \nu_p/\nu_\rho$. To obtain the result of Stinchcobmbe (1979a) we have to require now $\varphi(x) \sim x^{-1}$ for large x. This then will produce the following result for the DC conductivity exponent μ

$$\mu = \nu_p(d-2) + \zeta. \tag{4.54}$$

For $\delta s' \neq 0$, using result (4.53) we can write

$$\langle \sigma \rangle / D = (\delta p)^\mu f(\delta s(\delta p)^{-\nu_p/\nu_s}). \tag{4.55}$$

In the limit $\delta p \to 0$ for the conductivity to stay finite we must require $f(y) \sim y^x$ which gives $x = (\mu/\nu_p)\nu_s$ and

$$\langle v \rangle / D \propto s^{(\mu/\nu_p)\nu_s}. \tag{4.55a}$$

Considering now the diffusion coefficient in the limit $t \to \infty$, we have to distinguish between the cases $\delta p \neq 0$ and $\delta p = 0$. When $\delta p \neq 0$ we must have

$$D(t) = l^2 \tau_0^{-1} b^{2-1/\nu_s} f(\delta\rho b^{1/\nu_p}, \delta p b^{1/\nu_p}) \tag{4.56}$$

which, with use of constraint $\delta p b^{1/\nu_p} = 1$, gives $(v = 0)$

$$D(t) = l^2 \tau_0^{-1} (\delta p)^{-\nu_p(2-1/\nu_s)} \hat{\varphi}(\delta\rho(\delta p)^{-\nu_p/\nu_\rho}). \tag{4.57}$$

This result shoud be in agreement with Einstein's relation (2.8) which is possible only if $\hat{\varphi}(x) \sim x^\alpha$ for large x. This then produces

$$\lim_{\substack{t \to \infty \\ \delta p > 0}} D(t) \propto (\delta p)^{\mu - \beta_p} \tag{4.58}$$

with known percolation exponent β_p and the exponent α is found to be

$$\alpha = -\theta\nu_\rho + \nu_p(1/\nu_s - 2) \tag{4.59}$$

where $\theta = (\mu - \beta_p)/\nu_p$.

For $\delta p = 0$ using scaling property of the function $\hat{\varphi}$ we obtain, using the same technique as before

$$D(t) \propto t^{-\theta\nu_s}. \tag{4.60}$$

Because of the relation

$$\langle R^2(t) \rangle = \int_0^t d\tau \, D(\tau) \tag{4.61}$$

we obtain as well

$$\langle R^2 \rangle \propto t^{-\nu_s \theta + 1}. \tag{4.62}$$

This relation should be in accord with equation (4.26) which is formally valid for the case $\delta p = 0$. We are forced now, because of equation (4.61), to redefine the value of the exponent ω_D given by (4.27) which becomes

$$\tilde{\omega}_D = \nu_p (1/\nu_s - 2 - \theta). \tag{4.63}$$

From here we see that, if the identification $1/\nu_s = 2 + \theta$ is made, the well known single cluster results of Gefen *et al* (1983) are reproduced. Notice also that the relation $1/\nu_s = 2 + \theta$ *becomes exact at the fixed point for p* according to (4.26), (4.61) and (4.62). For $\delta p < 0$ we anticipate that $\langle R^2 \rangle$, given by (4.49), is time-independent. This gives

$$\langle R^2 \rangle \propto (\delta p)^{-2\nu_p} \tag{4.64}$$

when the relation $\nu_s^{-1} = 2 + \theta$ is used. Finally, consider the implications of scaling for $\langle R \rangle$ defined in (4.53). We have again to distinguish between the cases $\delta p \neq 0$ and $\delta p = 0$. For $\delta p > 0$ we should require $\langle R \rangle \propto t$ according to (3.12) and (3.14). Using (4.51) and putting there $\delta v b^{1/\nu_v} = 1$ we obtain

$$\langle R \rangle = l(\delta v)^{-\nu_v} \varphi(\delta p (\delta v)^{-\nu_v/\nu_p}, \, t(\delta v)^{\nu_v/\nu_s} \delta p (\delta v)^{-\nu_v/\nu_p})$$
$$= l(\delta v)^{-\nu_v} t((\delta v)^{\nu_v/\nu_s} \varphi_1(\delta p (\delta v)^{-\nu_v/\nu_p}, \, \delta p (\delta v)^{-\nu_v/\nu_p}). \tag{4.65}$$

Following the standard prescriptions (Domb and Lebowitz 1984), assuming the analyticity of φ_1, we can expand function φ_1 in power series in $x = \delta p^{-1} (\delta v)^{\nu_v/\nu_p}$ provided that δp is kept fixed. Then the leading term of this expansion will have the following form

$$\langle R \rangle \sim l(\delta v)^{-\nu_v(1-1/\nu_s)} (vp(\delta p)^{-\nu_v/\nu_p})^{\omega_1} t \tag{4.66}$$

with the exponent ω_1 obtained from the requirement that $\langle R \rangle$ should stay finite when $\delta v \to 0$. This gives the following result $\omega_1 = \nu_p(\nu_s^{-1} - 1)$. As will be demonstrated below on the two-dimensional example, the presence of bias does not affect the value of the exponent ν_s. Using an approximation $\nu_s^{-1} = 2 + \theta$ we obtain $\omega_1 = \nu_p(1 + \theta)$. This result, apparently, is valid only for the case of very strong bias. Consider finally the situation when $\delta p = 0$. Now we cannot demand $\langle R \rangle \propto t$. Going back to (4.51) we have

$$\langle R \rangle = lt^{\nu_s} f(\delta p t^{\nu_s/\nu_p}, \, \delta v t^{\nu_s/\nu_v}). \tag{4.67}$$

For small t using an approximation $\nu_s^{-1} = 2 + \theta$ we obtain the standard result $\langle R \rangle \propto t^{1/(2+\theta)}$ (Gefen *et al* 1983) but for the larger times the system tends to the new regime not accessible within the present renormalisation group treatment. The crossover time is defined according to

$$t_{\text{cross}} = \min((\delta v)^{-\nu_v/\nu_s}, \, (\delta p)^{-\nu_p/\nu_s}). \tag{4.68}$$

To determine the exponent ν_v as well as to demonstrate that the rest of the exponents remain the same, we provide the reader with very simple illustrative two-dimensional calculations.

5. Real space renormalisation of the random asymmetric hopping model ($\varepsilon \gg 1$). Two-dimensional case

In the preceeding sections we have already developed many components needed for the extension of the renormalisation group method to higher dimensions. Yet some of the questions still need to be discussed. For example, following Redner (1982), it is conceivable to assume that results of asymmetric hopping will depend upon the orientation of the external electric field with respect to axes of the hypercubic lattice. As was already mentioned in § 3, for the case of $\varepsilon \ll 1$, the renormalisation analysis will produce, in general, different results depending on presence or absence of angular averaging *in addition* to the impurity averaging. Here we shall restrict ourselves only to the case of impurity averaging so that we have to distinguish, according to Redner (1982), between the fully directed and the partially directed percolation. In fact, the above terminology is somewhat misleading for the following reason. For a given system the impurity averaging must be performed with or without the presence of the electric field, i.e., as far as the averaging process is concerned, we have still just, say, the standard bond percolation problem. This then implies that unlike the true case of directed percolation (Obukhov 1980), where we have to distinguish between the two percolation lengths $\xi_{p\parallel}$ and $\xi_{p\perp}$ in the case of asymmetric hopping we still have just one correlation length ξ_p because the distribution of bonds is independent of external electric field. What makes the problem anisotropic is anisotropic hopping rates. This then creates anisotropy *dynamically* and *not statically*. Despite all that the concept of fully directed against partially directed percolation turns out to be very useful and we would like now to make it more precise. In the case of fully directed percolation the electric field is oriented parallel to the diagonal $(1, 1, \ldots, 1)$ of the d-dimensional hypercube whereas in the case of partially directed percolation the electric field is oriented along a particular axis of the lattice so that perpendicular to this axis the system is isotropic. Here, for simplicity, we would like to consider only the two-dimensional case which already has all the features needed for further development.

5.1. *Real space renormalisation of the random asymmetric hopping model. Fully directed case*

For the two-dimensional case we would now like to develop some sort of Migdal-Kadanoff renormalisation scheme. Although it does not provide very accurate results it gives, nevertheless, the correct orders of magnitude needed for illustrative purposes. A very simple realisation of the above scheme for the static symmetric case was originally proposed by Stinchcombe (1979a, b). His method differs from the traditionally accepted (Burkhardt 1982) where we have to shift bonds *first* and *then* to perform a decimation. According to Stinchcombe we need *first* to decimate horizontal (vertical) bonds and *then* shift them. Because the Laplace variable s plays in the random hopping problem the same role as the constant external magnetic field, analogously to the magnetic case we encounter here the same kind of unpleasant problems upon the bond shifting (Jayaprakash *et al* 1978). To begin, we have to notice first that in the two-dimensional case the transfer matrix (4.7a) should be rewritten as follows

$$A_{\omega, \varepsilon, s}(i, i+1) = \exp[-\tfrac{1}{2}D(\varphi_i - \varphi_{i+1})^2 - \tfrac{1}{2}v\varphi_{i+1}(\varphi_i - \varphi_{i+1})] \exp(-\tfrac{1}{4}s\varphi_i^2 - \tfrac{1}{4}s\varphi_{i+1}^2). \tag{5.1}$$

This comes from the fact that for the square lattice the coordination number is 4 (to

be compared with 2 for the linear chain). Evidently, in any dimension for the lattice with coordination number x we should have instead of (5.1)

$$A_{\omega,\varepsilon,s}(i, i+1) = \exp[-\tfrac{1}{2}D(\varphi_i - \varphi_{i+1})^2 - \tfrac{1}{2}v\varphi_{i+1}(\varphi_i - \varphi_{i+1})]\exp[-(s/x)\varphi_i^2 - (s/x)\varphi_{i+1}^2].$$
$$(5.1a)$$

For the linear chain we used prescription (4.25) for computing the exponent z. It is clear that in the case of the pure chain result (4.25) can be obtained in two ways. First, we can combine the left and right recurrences for the bond in question or, second, we can combine the right recurrence for the given bond with the left recurrence for the subsequent bond. These two operations will produce identical results bringing the numerical prefactor in front of s back to one in agreement with (4.5). For the random case we shall adopt the same rule (4.25). In higher than one dimension we have to modify the rule (4.25) in a straightforward way because of the different coordination number of the lattice. The principle upon which the recurrence for s can be built is clear. Consider, for example, some point i on the square lattice which remains after the first decimation. Then we shall, in general, obtain four recurrences coming from four bonds attached to this point. We have to sum all four of them to restore the numerical prefactor in front of s back to unity. Generalisation to other lattices is straightforward. When the Migdal-Kadanoff method is used the situation is not so simple (Jayaprakash *et al* 1978). We adopt here a method which closely follows the method originally proposed by Stinchcombe (1979b). Here we generalise Stinchcombe's method to the case of time dependence in the same way as he had already done for the static symmetric hopping problem. Migdal's procedure for the fully directed case on a two-dimensional square lattice is depicted in figure 1. Following Stinchcombe (1983), it is also instructive to consider an auxiliary problem, depicted in figure 2, in order to have a better understanding as to how to treat the 'magnetic field' (in our case s variable). In the preceding section we have obtained all necessary formulae to accomplish the above task. Following Stinchcombe (1979b) we *first* decimate bonds combined in sequence and *then* join them in parallel.

The model example presented in figure 2 produces the following recurrence for the probability p

$$p' = 2p^2 - p^4 \qquad (5.2)$$

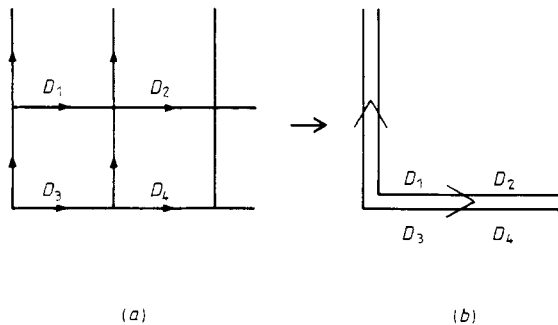which can be trivially obtained if we restrict ourselves with binary type of probability

Figure 1. Migdal's decimation procedure for the case of square lattice. Basic block (*a*) goes into (*b*) upon decimation *first* and bond shifting *second* as suggested by Stinchcombe (1979b).
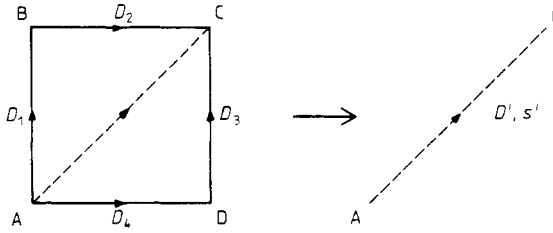
Figure 2. Simplest example of the renormalisation transformation which can be performed exactly. Here D_1-D_4 are in general random conductivities. The variable s associated with each bond before renormalisation is not shown explicitly but shoud be taken into account in order to obtain the recurrence for $s: s \rightarrow s'$.

distributions similar to (4.41) and (4.42). Indeed, we have in this case the following obvious relation $(1 - p^2)^2 = 1 - p'$ which immediately produces (5.2). Using these results we obtain the following recurrence for D

$$p'D' = p^4 \left(v + \frac{1}{2} \frac{(2D - v)^2}{(2D + 2s - v)} \right) + p^2(1 - p^2) \left(v + \frac{1}{2} \frac{(2D - v)^2}{(2D + 2s - v)} \right). \tag{5.3}$$

Similarly, we obtain the following recurrence for v

$$pv' = p^4 2v + 2p^2(1 - p^2)v. \tag{5.4}$$

Finally, for the s recurrence we obtain

$$\frac{s'}{2} = 2p^4 \left(s + D - \frac{v}{2} - \frac{1}{2} \frac{(2D - v)^2}{2(2D + 2s - v)} \right)$$

$$+ 2p^2(1 - p^2) \left(2s + D - \frac{v}{2} - \frac{1}{2} \frac{(2D - v)^2}{(2D + 2s - v)} \right) + 2s(1 - p^2)^2. \tag{5.5}$$

When D and v are zero we obtain $s' = 4s$ in accordance with the requirement of Jayaprakash *et al* (1978), which was mentioned in § 4, that the renormalisation should leave the variable s at the undecimated sites unaltered. When $v = 0$ we obtain the recursion relations for the symmetric case.

To understand better the meaning of the last recurrence we provide the reader with figure 3. Finally, figure 2, when it is understood in the topological sense, is converted into figure 1(b) for the case of Migdal's transformation. The recurrences obtained
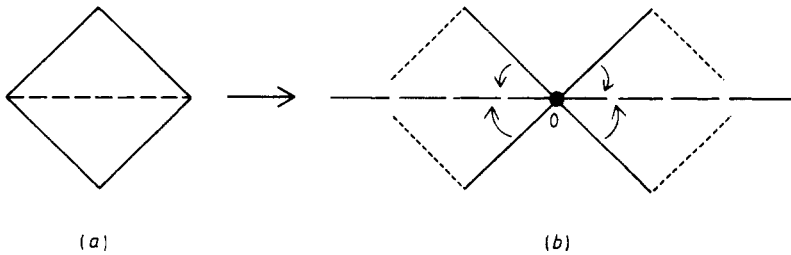

(a)   (b)

Figure 3. Basic building block (a) is converted into periodic array of such blocks (b). Each of four bonds outgoing from the point 0 in (b) participates in the decimation (point 0 is left untouched) which produces s recurrences. Then, what is left is combined in parallel so that we come back to the one-dimensional case when the rule (4.25) can be applied.

permit us to obtain the critical exponents. Because the recurrence (5.2) for the probability p is the same as that for the symmetric case we can write immediately $p^* = 0.618$, $\lambda_p = 1.528$ and $v_p = 1.635$. Now for the v recurrence we obtain

$$v' = 2vP^* \tag{5.6}$$

and, hence, $\lambda_v = 1.236, \nu_v = 3.27$. We shall consider for the s recurrence (5.5) the only one physically sensible fixed point $s^* = 0$ (i.e. $t \to \infty$). Then for the D recurrence we obtain

$$D^* = p^*(D^* + \tfrac{1}{2}v^*). \tag{5.7}$$

Analysis of this recurrence depends crucially on the choice of the fixed point for v. As it follows from (5.6), there are two fixed points for v: $v^* = 0$ and $v^* = \infty$. According to the philosophy of the renormalisation group analysis (Nelson and Fisher 1975) we have to choose fixed point $v^* = 0$ as physically relevant because it is unstable. Then we have for the recurrence (5.7) the same situation as for the isotropic case.

For the 'resistance' ρ, $\rho = 1/D$, we obtain $\nu_\rho = 1.44$. Using the above results we obtain for the s recurrence the following result

$$\tfrac{1}{4}s' = s(1 + p^{*2}) = 1.3819s \tag{5.8}$$

which surprisingly coincides with that which could be obtained for the symmetric case. This then produces immediately $\nu_s = 0.4054$ or $z = 2.467$. This accomplishes the task of computation of all relevant exponents in two dimensions.

6. Computation of the density of states

It is known that the density of states is related to the computation of the trace of the averaged propagator $\langle G_{n,m} \rangle_0$ (Feynman 1972). Using the fact that

$$\sum_n \langle G_{nn} \rangle_0 = \sum_m \frac{1}{s + \varepsilon_m} \tag{6.1}$$

and the formula $(x \mp i\gamma)^{-1} = P(1/x) \pm i\pi\delta(x)$ for $\gamma \to 0$ we obtain for the density of states the following known result

$$\rho(s) = \lim_{\gamma \to 0^+} \frac{\pi}{N} \text{Im} \left\langle \sum_n G_{nn}(s - i\gamma) \right\rangle_0 \tag{6.2}$$

where N is the number of lattice sites and s lies on the real axis. Using a d-dimensional generalisation of the expression (4.30) for the 'free energy' per site we obtain

$$\rho(s) = \lim_{\gamma \to 0^+} \frac{\pi}{N} \text{Im} \frac{\partial}{\partial s} \langle \ln Z(j = 0, s) \rangle_0 \big|_{s = s - i\gamma}. \tag{6.3}$$

During the procedure of real space renormalisation the number of lattice sites is reduced. Hence, N must be replaced by N' appropriate to the particular iteration step during the course of renormalisation (Rammal 1984). Using these results we can write for the free energy per site the following formal expression

$$\mathcal{F} = f(D, v, s, p) = N^{-1} \langle \ln Z(j = 0, D, v, s) \rangle_0. \tag{6.4}$$

Using the results of Nelson and Fisher (1975) we can now write

$$f(D', v', s', p') = b^d f(D, v, s, p).$$
(6.5)

In the scaling regime we obtain from here the following result

$$\mathcal{F} = b^{-d} f(\delta p b^{1/\nu_p}, s b^{1/\nu_s}, \delta v b^{1/\nu_v}, \delta p b^{1/\nu_p}).$$
(6.6)

Consider now the case $\delta p = 0$, i.e. the situation at the percolation threshold. In the absence of bias in the scaling regime of Gefen *et al* (1983) we obtain

$$\mathcal{F} \propto s^{d\nu_s}$$
(6.7)

while when the bias is present this result should be replaced by

$$\mathcal{F} \propto s^{d\nu_s} \varphi(\delta v s^{-\nu_s/\nu_v}).$$
(6.7a)

If we make an identification $\nu_s \approx (2 + \theta)^{-1}$ then for $v s^{-\nu_s/\nu_v} \le 1$ the phenomenological result of Alexander and Orbach (1982) holds. Indeed using the definition (6.3) combined with (6.7) (or (6.7a) for $v s^{-\nu_s/\nu_v} \le 1$) we obtain

$$\rho(s) \propto s^{d/(2+\theta)-1}$$
(6.8)

which coincides with the result predicted by Alexander and Orbach (1982) if the usual dimension d is replaced by the fractal dimension $\tilde{d}$.

7. Discussion

In the previous sections we have developed a rather crude renormalisation group method for studying the biased random walk on random lattice for the fully directed case using the terminology of Redner (1982). Evidently, more sophisticated treatments are possible as well and will be given in the future. The purpose of our work was to demonstrate that the simple phenomenological fractal picture of the 'ant in the labyrinth' problem developed by Alexander and Orbach (1982) and Gefen *et al* (1983) breaks down as long as some complications are introduced into the above picture. The presence of bias is only one of such complications considered here. As far as we can see, none of them could be overcome by just simple scaling arguments so that renormalisation group as well as other analytical methods should be used to explain more sophisticated situations. We hope that our approach will stimulate more elaborate calculations which will enable the diffusion-conduction problem to lie in the same 'universality class' as other critical phenomena problems.

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